

Physics 7: Analytical Mechanics

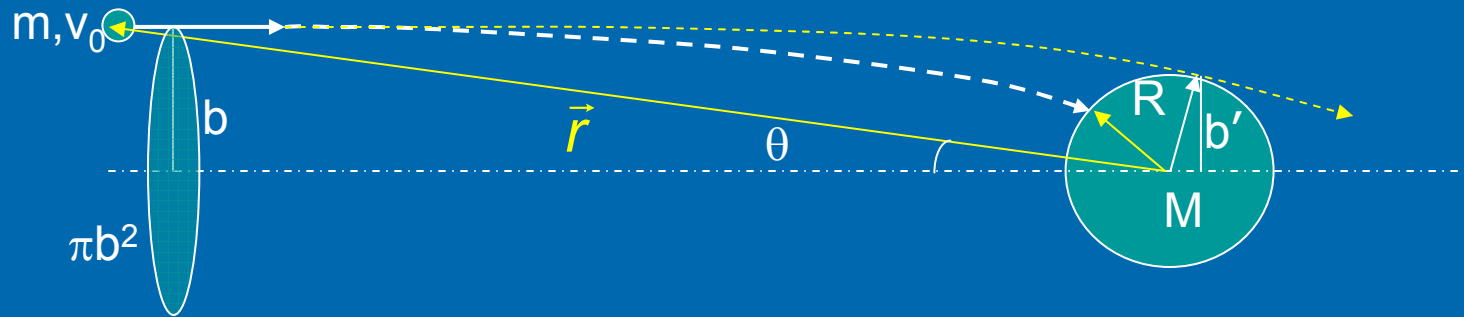
Lecture 16

Angular Momentum of a Solid Body

Greg Landsberg

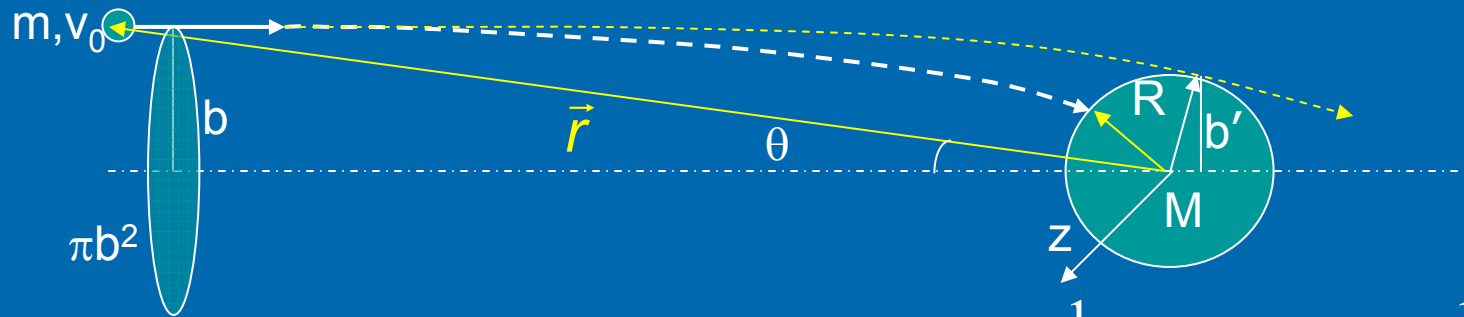
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Capture Cross Section of a Star



- Let's find the maximum *impact parameter* b for a remote asteroid of mass m moving towards a star of mass M and radius R with the velocity v_0 , for which it hits the star ("captured" by it) and the "capture cross section," $\sigma = \pi b^2$
- If there is no gravity, $b = R$ and $\sigma = \pi R^2$
- If there is gravitational attraction, $\sigma = \pi b^2 > \pi R^2$; here b is the initial impact parameter of a "grazing" trajectory
- Note that both energy and angular momentum are conserved (central force)

Capture Cross Section (cont'd)



- From energy conservation: $E = K + U = \frac{1}{2}mv^2 - mMG_N \frac{1}{r} = \text{const}$
- From angular momentum conservation:

$$\vec{L} = \vec{r} \times m\vec{v} = -mvr \sin\theta \hat{k} = \text{const}$$

- Initially, $r = +\infty$; $v = v_0$, $r \sin\theta = b \Rightarrow E = E_0 = \frac{1}{2}mv_0^2$; $L = L_0 = -mv_0 b$

- At the time of a grazing impact:
$$\begin{cases} L = -mv(R) \cdot R = -mv_0 b \\ E = \frac{1}{2}mv^2(R) - G_N Mm \frac{1}{R} = \frac{1}{2}mv_0^2 \end{cases} \Rightarrow$$

$$\begin{cases} v(R) = v_0 b/R \\ b^2 = R^2 \left(1 + \frac{2G_N M}{v_0^2 R} \right) \end{cases} \Rightarrow \sigma = \pi R^2 \left(1 + \frac{G_N Mm/R}{mv_0^2/2} \right) = \pi R^2 \left(1 - \frac{U(R)}{E} \right)$$

Torque Due to Gravity

- For a rigid body, the torque due to the force of gravity is given by:

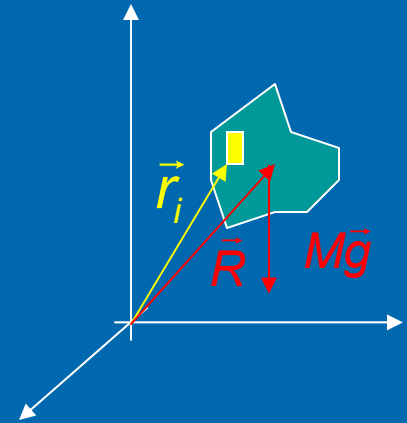
$$\vec{\tau} = \vec{R} \times \vec{W}$$

- Here \vec{R} is the position vector of the c.o.m. and \vec{W} is the total weight of the body: $\vec{W} = M\vec{g}$

- Let's prove it:

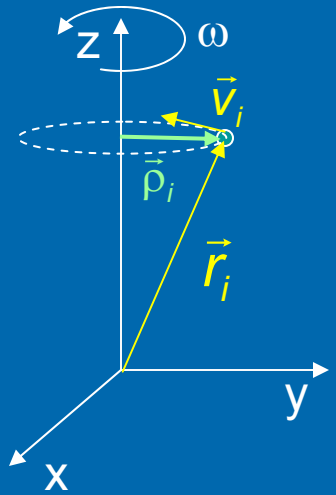
$$\vec{\tau}_i = m_i \vec{r}_i \times \vec{g} \Rightarrow \vec{\tau} = \sum_i m_i \vec{r}_i \times \vec{g} = \left(\sum_i m_i \vec{r}_i \right) \times \vec{g} =$$

$$M\vec{R} \times \vec{g} = \vec{R} \times \vec{W}$$



Angular Momentum and Fixed-Axis Rotation

- A particularly important practical case of rotational motion is a rotation about a fixed axis (i.e., the direction of an axis is fixed, but it can be translated, e.g. a car wheel)
- When a rigid body rotates about an axis, every particle remains at the same distance from the axis
- Let's choose a coordinate system with the point of origin on the axis and axis z being the axis of rotation
 - In this system, $|\vec{r}_i| = \text{const}$
 - This is only possible if $\vec{v}_i \perp \vec{r}_i$, which implies $|\vec{v}_i| = |\dot{\vec{r}}_i| = \omega \rho_i$
 - Angular momentum: $\vec{L}_i = \vec{r}_i \times m_i \vec{v}_i = L_z^i \hat{k} + \vec{L}_\perp$, where $L_z^i = m_i v_i \rho_i = m_i \rho_i^2 \omega$
 - Hence, $L_z = \sum_i L_z^i = \sum_i m_i \rho_i^2 \omega = I \omega$, where



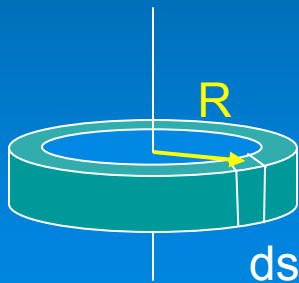
$$I \equiv \sum_i m_i \rho_i^2 - \text{moment of inertia}$$

Moment of Inertia for Simple Objects

- **Example 1:** Moment of inertia of a point-mass
 - $I = m\rho^2$; $L = I\omega = m\omega\rho^2$, where ρ is the distance from the axis
- **Note:** for continuously-distributed matter, the sum is replaced by an integral:

$$I \equiv \sum_i m_i \rho_i^2 \rightarrow \int \rho^2 dm = \int (x^2 + y^2) dm = \int (x^2 + y^2) \rho(x, y, z) dV$$

- **Example 2:** Moment of inertia of a hoop
 - Let's consider a uniform hoop of mass M and radius R , rotating about its geometrical center ("bicycle wheel"):



$$I = \int \rho^2 dm = \int_0^{2\pi R} R^2 \lambda ds, \text{ where } \lambda = M/2\pi R$$

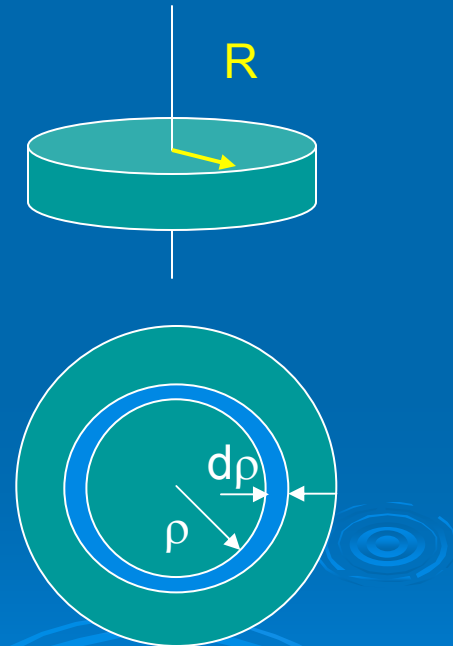
$$I = \int_0^{2\pi R} R^2 \frac{M}{2\pi R} ds = \frac{MR}{2\pi} 2\pi R = MR^2$$

Moment of Inertia for Simple Objects (cont'd)

➤ Example 3: Moment of inertia of a uniform disk

- Let's consider a uniform disk of mass M and radius R , rotating about its geometrical center
- Let's divide the disk into concentric hoops of radius ρ and thickness $d\rho$
- The area of this disk $dA = 2\pi\rho d\rho$ and hence the mass $dm = MdA/A = 2M\pi\rho d\rho/\pi R^2$
- The moment of inertia of this ring is $dI = dmp^2 = 2M\rho^3 d\rho/R^2$, and hence the total moment of inertia

$$I = \int dI = \int_0^R \frac{2M}{R^2} \rho^3 d\rho = \frac{2M}{R^2} \frac{R^4}{4} = \frac{MR^2}{2}$$



Moment of Inertia for Simple Objects (cont'd)

➤ Example 4: Moment of inertia of a stick

- Let's consider a stick of length L rotating about the axis perpendicular to it and going through its center



$$I = \int_{-L/2}^{L/2} x^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx = \frac{M}{L} \frac{x^3}{3} \Big|_{-L/2}^{L/2} = \frac{M}{3L} 2 \frac{L^3}{8} = \frac{ML^2}{12}$$

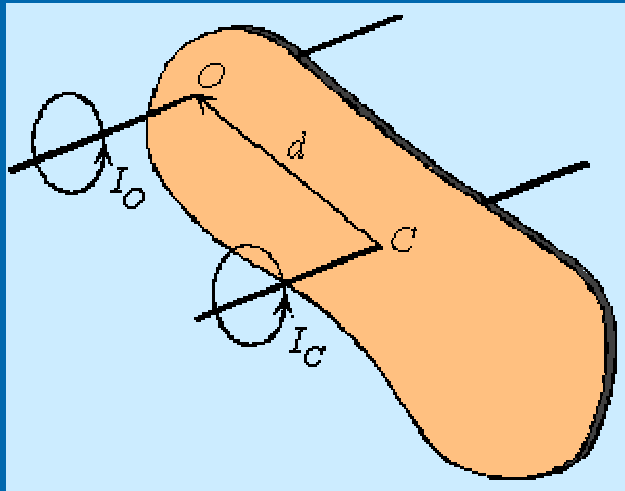
➤ Example 5: The same, except that the axis is at the end

$$I = \int_0^L x^2 dm = \int_0^L x^2 \frac{M}{L} dx = \frac{M}{L} \frac{x^3}{3} \Big|_0^L = \frac{ML^2}{3}$$

➤ Example 6: A uniform sphere of radius R and mass M

$$I = \frac{2MR^2}{5} \text{ (it's one of your homework assignments)}$$

The Parallel Axis Theorem



- Let's assume that we know the moment of inertia of a body relative to the axis going through its c.o.m.; can we deduce moment of inertia about a parallel axis at a distance d from the c.o.m.?
- Let's set a coordinate frame with the axis z along the new axis of rotation; then the position of a small element of volume is:

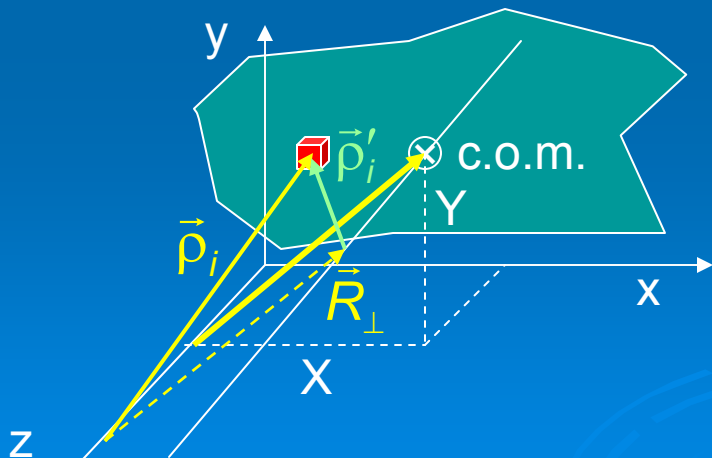
$$\vec{\rho}_i = x_i \hat{i} + y_i \hat{j} \quad \text{and} \quad I = \sum_i m_i \rho_i^2$$

- If the position of the c.o.m. is (X, Y, Z) , the vector \perp to the z -axis that connects it to the c.o.m. is

$$\vec{R}_\perp = X \hat{i} + Y \hat{j} \quad \text{and} \quad \vec{\rho}_i = \vec{\rho}'_i + \vec{R}_\perp$$

- The moment of inertia about the c.o.m. axis is:

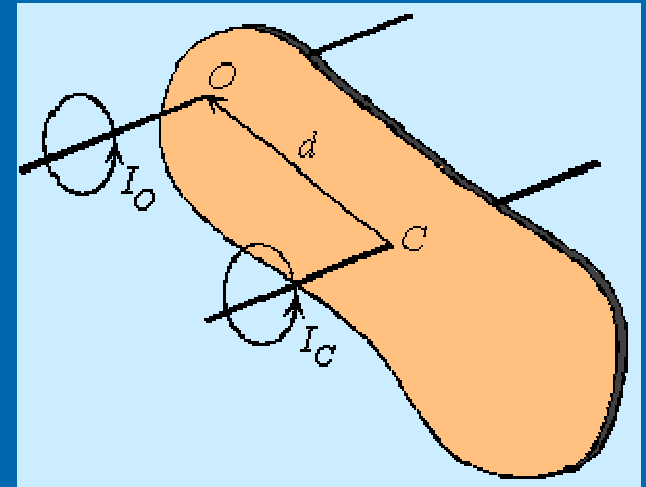
$$I = \sum_i m_i \rho_i^2 = \sum_i m_i (\vec{\rho}'_i + \vec{R}_\perp)^2$$



The Parallel Axis Theorem (cont'd)

- We get:

$$\begin{aligned} I &= \sum_i m_i \rho_i^2 = \sum_i m_i (\vec{\rho}'_i + \vec{R}_\perp)^2 = \\ &= \sum_i m_i \rho_i'^2 + 2\vec{R}_\perp \cdot \sum_i m_i \vec{\rho}'_i + M |\vec{R}_\perp|^2 = \\ &= I_0 + Md^2 + 2\vec{R}_\perp \cdot \sum_i m_i (\vec{\rho}_i - \vec{R}_\perp) = \\ &= I_0 + Md^2 + 2\vec{R}_\perp \cdot [M\vec{R} - M\vec{R}_\perp] = \\ &= I_0 + Md^2 + 2\vec{R}_\perp \cdot M\vec{R}_\parallel = I_0 + Md^2 \end{aligned}$$



- Thus, we proved the parallel axis theorem: **the moment of inertia about an axis at a distance d from the c.o.m. is equal to the moment of inertia about a parallel axis going through the c.o.m. I_0 plus Md^2**

$$I = I_0 + Md^2$$

Examples of Parallel Axes

- **Example 1:** Find the moment of inertia of a rod rotated about one of its ends

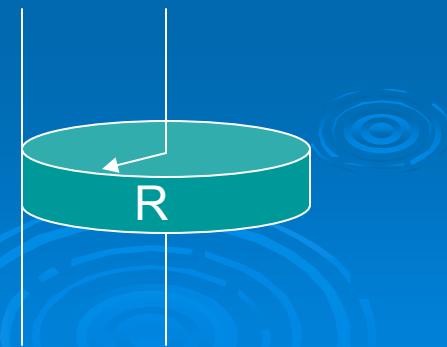
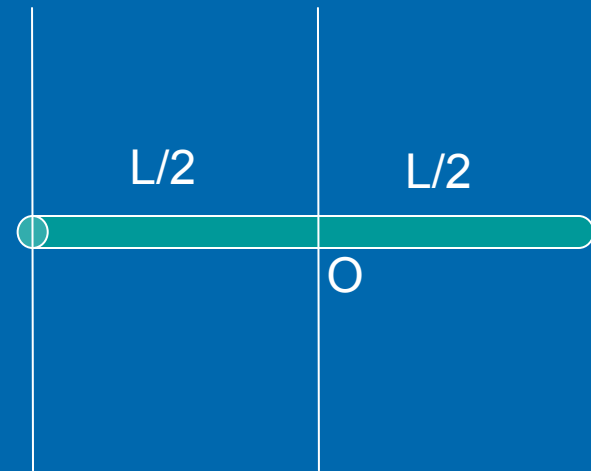
- Solution:

$$I = I_0 + M(L/2)^2 = ML^2/12 + ML^2/4 = (3+1) ML^2/12 = \mathbf{ML^2/3}$$
 – same as we have derived explicitly before

- **Example 2:** Find the moment of inertia of a disk rotated about an axis going through its side

- Solution:

$$I = I_0 + MR^2 = MR^2/2 + MR^2 = \mathbf{3MR^2/2}$$



Dynamics of Pure Rotation About an Axis

- It's tempting to break the torque into an external and internal
 - Unfortunately, there is no way to prove from the third Newton's law that the total internal torque is equal to 0, but it's an experimental fact, which we will take for granted for now
 - Hence, change in the angular momentum is due to the external torque

- Let's now consider pure rotation about axis z:

$$L_z = I\omega$$

- We have:

$$\vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt} \Rightarrow \tau_z = \frac{d}{dt}(I\omega) = I \frac{d\omega}{dt} = I\alpha,$$

where α is angular acceleration:

$$\alpha = \dot{\omega} = \ddot{\theta}$$

- Since the axis is fixed, we will drop index z: $\tau = I\alpha$
- Let's find the kinetic energy of the rotating body:

$$K = \frac{1}{2} \sum_i m_i v_i^2 = \frac{1}{2} \sum_i m_i (\omega \rho_i)^2 =$$

$$\frac{\omega^2}{2} \sum_i m_i \rho_i^2 = \frac{I\omega^2}{2}$$

Announcements

- **Homework #9 has been posted**
- Second midterm will be delayed till Friday, November 19
 - Three problems on (non-)conservation of energy, momentum, and angular momentum